

On Torsion-free Vacuum Solutions of the Model of de Sitter Gauge Theory of Gravity

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(Dated: November 2007)

It is shown that all vacuum solutions of Einstein field equation with a positive cosmological constant are the solutions of a model of dS gauge theory of gravity. Therefore, the model is expected to pass the observational tests on the scale of solar system and explain the indirect evidence of gravitational wave from the binary pulsars PSR1913+16.

PACS numbers: 04.50.+h, 04.20.Jb

The astronomical observations show that our universe is probably an asymptotically de Sitter(dS) one [1, 2]. It arises the interests on dS gauge theories of gravity. There is a model of the dS gauge theory of gravity, which was first proposed in the 1970's [3, 4]. The model can be stimulated from dS invariant special relativity [5, 6, 7] and the principle of localization [8], just like that the Poincaré gauge theory of gravity may be stimulated from the Einstein special relativity and the localization of Poincaré symmetry [9]. The principle of localization is that the full symmetry of the special relativity as well as the laws of dynamics are both localized. The gravitational action of the model takes the Yang-Mills form of [3, 4, 8]

$$S_{\text{GYM}} = \frac{1}{4g^2} \int_{\mathcal{M}} d^4x e \text{Tr}_{\text{dS}}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}), \quad (1)$$

where $e = \det(e_\mu^a)$ is the determinant of the tetrad e_μ^a , g is a dimensionless coupling constant introduced as usual in the gauge theory to describe the self-interaction of the gauge field,

$$\mathcal{F}_{\mu\nu} = (\mathcal{F}^{AB}_{\mu\nu}) = \begin{pmatrix} F^{ab}_{\mu\nu} + R^{-2} e^{ab}_{\mu\nu} & R^{-1} T^a_{\mu\nu} \\ -R^{-1} T^b_{\mu\nu} & 0 \end{pmatrix} \quad (2)$$

is the curvature of dS connection¹

$$\mathcal{B}_\mu = (\mathcal{B}^{AB}_\mu) = \begin{pmatrix} B^{ab}_\mu & R^{-1} e^a_\mu \\ -R^{-1} e^b_\mu & 0 \end{pmatrix} \in so(1, 4), \quad (3)$$

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¹ The same dS-connection with different dynamics has also been explored in Ref. [10].

and \mathbf{Tr}_{dS} is the trace for the $so(1, 4)$ indices A, B . In Eq.(2), $F_{\mu\nu}^{ab}$ and $T_{\mu\nu}^a$ are the curvature and torsion tensors of the Lorentz connection $B_{b\mu}^a \in so(1, 3)$, respectively, R is the dS radius, and $e_{ab}^{\mu\nu} = e_a^\mu e_b^\nu - e_a^\nu e_b^\mu$. In terms of $F_{\mu\nu}^{ab}$ and $T_{\mu\nu}^a$, the gravitational action can be rewritten as

$$S_{\text{GYM}} = - \int_{\mathcal{M}} d^4x e \left[\frac{1}{4g^2} F_{\mu\nu}^{ab} F_{ab}^{\mu\nu} - \chi(F - 2\Lambda) - \frac{\chi}{2} T_{\mu\nu}^a T_a^{\mu\nu} \right], \quad (4)$$

where $F = \frac{1}{2} F_{\mu\nu}^{ab} e_{ab}^{\mu\nu}$ the scalar curvature, the same as the action in the Einstein-Cartan theory, $\chi = 1/(16\pi G)$ is a dimensional coupling constant, $\Lambda = 3/R^2 = 3\chi g^2$ is the cosmological constant.

The gravitational field equations, obtained by the variation of the total action

$$S_{\text{T}} = S_{\text{GYM}} + S_{\text{M}} \quad (5)$$

with respect to e_μ^a, B_{μ}^{ab} , are

$$T_a^{\mu\nu}{}_{||\nu} - F_a^\mu + \frac{1}{2} F e_a^\mu - \Lambda e_a^\mu = 8\pi G (T_{\text{Ma}}^\mu + T_{\text{Ga}}^\mu), \quad (6)$$

$$F_{ab}^{\mu\nu}{}_{||\nu} = R^{-2} (16\pi G S_{\text{Mab}}^\mu + S_{\text{Gab}}^\mu). \quad (7)$$

Here, S_{M} is the action of the matter source with minimum coupling, $||$ represents the covariant derivative using both Christoffel symbol $\{\mu_{\nu\kappa}\}$ and connection $B_{b\mu}^a$, $F_a^\mu = -F_{ab}^{\mu\nu} e_\nu^b$,

$$T_{\text{Ma}}^\mu := -\frac{1}{e} \frac{\delta S_{\text{M}}}{\delta e_\mu^a} \quad (8)$$

$$T_{\text{Ga}}^\mu := g^{-2} T_{\text{Fa}}^\mu + 2\chi T_{\text{Ta}}^\mu \quad (9)$$

are the tetrad form of the stress-energy tensor for matter and gravity, respectively, where

$$\begin{aligned} T_{\text{Fa}}^\mu &:= -\frac{1}{4e} \frac{\delta}{\delta e_\mu^a} \int d^4x e \text{Tr}(F_{\nu\kappa} F^{\nu\kappa}) \\ &= e_a^\kappa \text{Tr}(F^{\mu\lambda} F_{\kappa\lambda}) - \frac{1}{4} e_a^\mu \text{Tr}(F^{\lambda\sigma} F_{\lambda\sigma}) \end{aligned} \quad (10)$$

is the tetrad form of the stress-energy tensor for curvature and

$$\begin{aligned} T_{\text{Ta}}^\mu &:= -\frac{1}{4e} \frac{\delta}{\delta e_\mu^a} \int d^4x e T_{\nu\kappa}^b T_b^{\nu\kappa} + T_a^{\mu\nu}{}_{||\nu} \\ &= e_a^\kappa T_b^{\mu\lambda} T_{\kappa\lambda}^b - \frac{1}{4} e_a^\mu T_b^{\lambda\sigma} T_{\lambda\sigma}^b \end{aligned} \quad (11)$$

the tetrad form of the stress-energy tensor for torsion, and

$$S_{\text{Mab}}^\mu = \frac{1}{2\sqrt{-g}} \frac{\delta S_{\text{M}}}{\delta B_{\mu}^{ab}} \quad (12)$$

and S_{Gab}^μ are spin currents for matter and gravity, respectively. Especially, the spin current for gravity can be divided into two parts,

$$S_{\text{Gab}}^\mu = S_{\text{Fab}}^\mu + 2S_{\text{Ta}b}^\mu, \quad (13)$$

is used [13], where

$$E_{\mu\nu\kappa\lambda} = \frac{1}{2}(g_{\mu\kappa}S_{\nu\lambda} + g_{\nu\lambda}S_{\mu\kappa} - g_{\mu\lambda}S_{\nu\kappa} - g_{\nu\kappa}S_{\mu\lambda}), \quad (20)$$

$$G_{\mu\nu\kappa\lambda} = \frac{\mathcal{R}}{12}(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}), \quad (21)$$

$$S_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4}\mathcal{R}g_{\mu\nu}. \quad (22)$$

On the other hand, the vacuum Einstein field equation with a (positive) cosmological constant reads

$$\mathcal{R}^\mu{}_\nu - \frac{1}{2}\mathcal{R}\delta^\mu_\nu + \Lambda\delta^\mu_\nu = 0. \quad (23)$$

It results in

$$\mathcal{R} = 4\Lambda, \quad \mathcal{R}^\mu{}_\nu = \Lambda\delta^\mu_\nu, \quad (24)$$

and thus

$$S_{\mu\nu} = 0. \quad (25)$$

Since the Weyl tensor is totally traceless, the stress-energy tensor for Riemann curvature vanishes, *i.e.*,

$$T_{R\mu}{}^\nu = 0. \quad (26)$$

Therefore, all vacuum solutions of Einstein field equation with a cosmological constant are solutions of Eq.(16). In addition, the Bianchi identity

$$\mathcal{R}^{\mu\nu}{}_{\lambda\sigma;\kappa} + \mathcal{R}^{\mu\nu}{}_{\kappa\lambda;\sigma} + \mathcal{R}^{\mu\nu}{}_{\sigma\kappa;\lambda} = 0 \quad (27)$$

leads to

$$0 = \mathcal{R}^{\mu\nu}{}_{\lambda\sigma;\nu} - \mathcal{R}^\mu{}_{\lambda;\sigma} + \mathcal{R}^\mu{}_{\sigma;\lambda} = \mathcal{R}_{\lambda\sigma}{}^{\mu\nu}{}_{;\nu}. \quad (28)$$

Namely, Yang equation (17) is also satisfied. (The last step of Eq.(28) is valid because of Eq.(24).)

Therefore, we come to the conclusion that all vacuum solutions of the Einstein field equation with a positive cosmological constant are the torsion-free vacuum solutions of the model of dS gauge theory of gravity. In particular, the dS, Schwarzschild-dS, and Kerr-de Sitter metrics satisfy the Eqs.(6) and (7). Note that the Birkhoff theorem has been proved for the gravitational theory (4) without a cosmological constant [14]. So, the model is expected to pass the observational tests on the scale of solar system. In addition, the model has the same metric waves as general relativity and thus is expected to explain the indirect evidence of the existence of gravitational wave from the observation data on the binary pulsar PSR1913+16.

One might think that the above results are trivial because the Yang equation does not appear at all if the torsion-free condition is assumed in the action, in which case the tetrad and connection are not independent. However, the torsion-free manifold is just the specific situation of the the model. There is no reason to set the torsion to be zero before the variation.

In fact, it can be shown that all solutions of vacuum Einstein field equation with a positive cosmological constant are also the vacuum, torsion-free solutions of the field equations when the terms

$$F_a{}^\mu F_\mu{}^a, e_\nu^a e_\mu^b F_a{}^\mu F_b{}^\nu, e^{ab}{}_{\lambda\sigma} e^{cd}{}_{\mu\nu} F_{ab}{}^{\mu\nu} F_{cd}{}^{\lambda\sigma}, e_\sigma^b e_\mu^c F_{ab}{}^\mu F_{ac}{}^{\nu\sigma}, e_a^\lambda e_b^\sigma T_{\mu\lambda}^a T^{b\mu}_\sigma, e_a^\sigma e_b^\mu T_{\mu\lambda}^a T^{b\lambda}_\sigma$$

are added in the gravitational Lagrangian. Obviously, the last two terms have no contribution to the vacuum, torsion-free field equations, while the middle two terms contribute the same as the term $F_{ab}{}^{\mu\nu} F_{\mu\nu}{}^{ab}$ does thus only alter the unimportant coefficients. The first two terms add the term $(R_{[a}^\mu e_{b]}^\nu)_{;\nu}$ in Yang equation and the stress-energy tensor $R_{\mu\lambda} R^{\nu\lambda} - \frac{1}{4} \delta_\mu^\nu R_{\sigma\lambda} R^{\sigma\lambda}$ in Einstein equation. Both of them vanish for the solutions of the vacuum Einstein equation with a positive cosmological constant.

Obviously, the conclusion is still valid if the integral of the second Chern form of the dS connection over the manifold is added in the action. Finally, the similar discussions can be applied to the AdS case as well.

Acknowledgments

We thank Z. Xu, B. Zhou and H.-Q. Zhang for useful discussions. This work is supported by NSFC under Grant Nos. 90503002, 10605005, 10775140, 10705048, 10731080 and Knowledge Innovation Funds of CAS (KJCX3-SYW-S03).

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